

# A Note on Forward and Backward Partial Differential Equations for Derivative Contracts with Forwards as Underlyings

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## Abstract

In this note we show that the forward partial differential equation for the valuation of European options proposed by Dumas, Fleming, and Whaley (DFW) does not hold for their general model with time and state-dependent local volatilities. To clarify things we analyze the forward and backward partial differential equations for contingent claims in detail. We consider the three cases of spot options on the spot, and spot and forward options on the forward. We find that the DFW equation is only valid when either the interest rate is equal to the dividend yield, or when the local volatility of the underlying is a function of time only. Furthermore, we show that in the general case of arbitrary (but deterministic) interest rates and dividend yields, the stock price has to be the argument of the local volatility function in the backward equation for the forward price of a claim on the forward, not the forward price.

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# 1 Introduction

Both Kolmogorow's forward and backward partial differential equation (pde) are widely used in the theory of option pricing. Recently, especially the forward equation has received much attention in the context of derivative pricing under a volatility smile (see, e.g., Andersen and Brotherton-Ratcliff (AB) [1], Dumas, Fleming, and Whaley [4], or Dupire [5]). Whereas the standard backward equation uses the stock price and calendar time as variables in the pricing function, the forward equation is based on the maturity date and the strike price of a standard European option, and it is furthermore only valid for this rather limited class of securities. The backward equation on the other hand holds for any underlying and any contingent claim and is thus far more general. Using the forward equation, however, allows the simultaneous valuation of more than one European option, which is a special advantage when finite difference methods have to be employed to solve the pde numerically.

In both theoretical analyses and empirical applications of derivative pricing models, it is sometimes convenient to use forward prices instead of spot prices. This transformation can be performed for both the underlying of a contingent claim and the claim itself. In the extreme case we could thus move from a standard spot option on the stock to a forward option on the forward written on the stock. One might think that this just involves a proper transformation of the coordinates in a given pde, but it is not in general true that the transition from spot prices to forward prices automatically removes all terms depending on interest rates and dividend yields. As we will show below, this technical trick only yields convenient expressions for the pde when we work with the backward equation. In the case of a forward pde we cannot in general obtain a comparably simple pde based on the forward prices of the underlying and the option, except when either the interest rate is equal to the dividend yield, i.e. the forward price is equal to the current stock price, or the local volatility of the underlying depends on time only. Since DFW [4] assume neither of these scenarios, a key result of our analysis is that the pde suggested in their paper is incorrect for their general model.

The remainder of the paper is organized as follows. In section 2 we will briefly summarize the theory on forward and backward pde's for contingent claim valuation. Section 3 contains the main results of the paper, namely the forward and backward equations for all four scenarios listed above, i.e. for both spot and forward options on both the underlying and the forward contract. Some conclusions are presented in section 4.

## 2 Forward and Backward Equations

### 2.1 The General Setting

Let  $(\Omega, \mathcal{F}, P)$  be a filtered probability space. The dynamics of the underlying stock under the equivalent risk-neutral probability measure  $\hat{P}$  are given by

$$dS_t = (r - \nu)S_t dt + \sigma(S, t)S_t d\widehat{W}_t, \quad (1)$$

where  $r$  and  $\nu$  stand for the risk-free rate and the continuous payout rate of the underlying, respectively. Both  $r$  and  $\nu$  are assumed to be constant. This assumption could easily be relaxed to make  $r$  and  $\nu$  functions of time. However, the gain in generality is rather small compared to the cost of a much more cumbersome notation.  $d\widehat{W}_t$  represents the increment of a standard one-dimensional Brownian motion, and  $\sigma(S, t)$  is called the *local volatility function*.

Besides the stock there is a second security called the money market account with price  $B_t$  at time  $t$  and dynamics given by

$$dB_t = B_t r dt \quad (2)$$

which, given  $B_0 = 1$ , yields  $B_t = e^{rt}$ .

Before going into the details we want to point out that there are two slightly different meanings of the terms 'backward equation' and 'forward equation'. From the mathematical point of view backward and forward equations are known as the Kolmogorow equations for the transition densities. From the economic point of view there also exists a so called *backward (pricing) equation* for an almost arbitrary contingent claim  $g(S, t)$  on  $S$ . This equation can be derived either by using duplication strategies or by integrating the transition density function. For the special case of plain vanilla options there also exists a so called *forward (pricing) equation*, which makes use of the special payoff structure for these contracts. With the Kolmogorow forward equation for the transition density one can easily derive the forward equation for a European call in the variables  $K$  and  $T$ . This equation is of great empirical interest because data of option prices vary in  $K$  and  $T$  rather than in  $S$  and  $t$ .

### 2.2 Backward Equation

Let  $C(S, t; K, T)$  denote the value at time  $t$  of a standard European call option with strike price equal to  $K$  and maturity date  $T$ , when the current stock price is equal to  $S$ .

Using standard no-arbitrage arguments, it can be shown that, given (1) and (2), the value of the option  $C = C(S, t; K, T)$  satisfies the fundamental Black-Scholes pde

$$rC = C_t + S(r - \nu)C_S + \frac{1}{2}S^2\sigma^2(S, t)C_{SS} \quad (3)$$

where subscripts denote partial derivatives. This equation has to be solved under the boundary condition

$$C(S, T; K, T) = g(S) \equiv \max(S - K, 0).$$

Note that in (3)  $K$  and  $T$  are held fixed, and the fundamental pde is the backward equation in the so called *backward variables*  $S$  and  $t$ . This equation holds in general for any claim with an (almost) arbitrary boundary condition.

An analogous backward equation holds for the transition density. Let  $p = p(S, t; y, T)$  denote the risk-neutral transition density for the stock price, i.e.  $p(S, t; \cdot, T)$  is the density of the process at time  $T$  when started at  $S$  at time  $t$ . Then this density satisfies the pde

$$p_t + S(r - \nu)p_S + \frac{1}{2}S^2\sigma^2(S, t)p_{SS} = 0 \quad (4)$$

subject to the boundary condition

$$p(S, T; y, T) = \delta(S - y),$$

where  $\delta(x)$  represents the Dirac delta function (see Shreve [7], p. 180).

## 2.3 Forward Equation

Under suitable regularity conditions on  $\sigma$  there also exists a pde in the so called *forward variables*  $y, T$  of the density  $p = p(S, t; y, T)$ , which is given by

$$p_T + (r - \nu)\frac{\partial}{\partial y}[yp] - \frac{1}{2}\frac{\partial^2}{\partial y^2}[\sigma^2(y, t)y^2p] = 0 \quad (5)$$

with boundary condition

$$p(S, t; y, t) = \delta(S - y).$$

A heuristic derivation of this result can be found in Shreve [7], pp. 306-08.

From the forward pde for the density we can derive a forward pde for plain vanilla options (see AB [1]). Using the fact that

$$\frac{\partial^2 C(S, t; K, T)}{\partial K^2} = e^{-r(T-t)}p(S, t; K, T),$$

and the forward pde for the density, we obtain after two integrations with respect to  $K$  the pde

$$C_T + (r - \nu)KC_K + \nu C - \frac{1}{2}\sigma^2(K, T)K^2C_{KK} = 0, \quad (6)$$

which has to be solved under the boundary condition

$$C(S, t; K, t) = \max(S - K; 0).$$

Note that the forward equation is only valid for a standard call, in contrast to the backward equation that holds for arbitrary claims.

### 3 Forward-Based Derivation of Backward and Forward Partial Differential Equations

#### 3.1 Options on Forwards

We now consider options on forwards. Let  $T_1$  and  $T_2$  be the maturity dates of the call and the underlying forward  $F$ , respectively. We denote the price of the call at time  $t$  with strike  $K$  on the forward by  $\bar{C}(F, t, T_2; K, T_1)$ . Assuming constant interest rates and dividend yields, the price of a forward with maturity  $T_2$  on the stock is given by  $F(S, t, T_2) = Se^{(r-\nu)(T_2-t)}$ . In the special case, when the maturities of the option and the forward contract are the same, i.e.  $T_1 = T_2 = T$ , the price of a call on the underlying stock is equal to the price of a call on the forward. At time  $T$ , both options have the same payoff, since at maturity the forward price converges to the spot, and both options do not have any cash flows between  $t$  and  $T$ . So it must be true that

$$C(S, t; K, T) = \bar{C}(F, t; T, K, T) \equiv \bar{C}(F, t; K, T),$$

which means we can interpret the transition from  $C$  to  $\bar{C}$  as a simple transformation of coordinates from  $S$  to  $F$ . For the general case, prices are given in Proposition 1 in Appendix A.

#### 3.2 Forwards on Options on Forwards

The ultimate goal is to find the forward price of a call on a forward contract (written on the underlying).

Having obtained the function  $\bar{C}$  as shown above, the forward price for delivery at time  $T_0$  ( $T_0 \leq T_1 \leq T_2$ ) of this option is given by

$$D(\bar{C}, t, T_0) \equiv D(F, t, T_2; K, T_1) = e^{r(T_0-t)} \bar{C}(F, t, T_2, K, T_1).$$

In what follows we will only consider the special case  $T_0 = T_1 = T_2 = T$ :

$$\begin{aligned} D(F, t, T; K, T) &\equiv D(F, t; K, T) \\ &= e^{r(T-t)} \bar{C}(F, t; K, T) \\ &= e^{r(T-t)} C(S, t; K, T), \end{aligned}$$

i.e., the forward price of the option is simply equal to the price of the option on the stock times the cost of carry.

### 3.3 The Backward Equation for Forward Contingent Claims on Forwards

From the Black-Scholes backward pde (3) for the claim  $C(S, t; K, T)$  in the backward variables  $S, t$  with  $K, T$  fixed we obtain the following forward-based backward pde of the function  $D = D(F, t; K, T)$

$$D_t + \frac{1}{2}\sigma^2(Fe^{-(r-\nu)(T-t)}, t)F^2D_{FF} = 0 \quad (7)$$

using the change of variables

$$F(S, t, T) = Se^{(r-\nu)(T-t)} \quad (8)$$

$$C(S, t; K, T) = D(Se^{(r-\nu)(T-t)}, t; K, T)e^{-r(T-t)} = D(F, t; K, T)e^{-r(T-t)}. \quad (9)$$

To see that (7) actually holds, compute the partial derivatives

$$\begin{aligned} C_t &= \frac{\partial}{\partial t} \left[ D(Se^{(r-\nu)(T-t)}, t; K, T)e^{-r(T-t)} \right] = e^{-r(T-t)} (rD - (r-\nu)FD_F + D_t) \\ C_S &= e^{-r(T-t)} \frac{\partial}{\partial F} D(F, t; K, T) \frac{\partial}{\partial S} F(S, t, T) = e^{-\nu(T-t)} D_F \\ C_{SS} &= e^{r(T-t)} e^{-2\nu(T-t)} D_{FF} \end{aligned}$$

and plug the results into the standard backward pde (3) to obtain (7). Note that the first argument of the local volatility function in (7) is the *discounted* forward price (i.e. the stock price), and not the forward price itself as indicated in the backward pde in DFW [4] (equation (4), p. 2087)<sup>1</sup> So when implementing their approach on a finite difference grid, one has to make sure to specify the local volatility function properly.

### 3.4 The Forward Equation for Forward Calls on Forwards

We will now derive the forward pde for a forward call on a forward. DFW [4] (equation (5), p. 2088) state the following forward pde for the forward price of a standard European option:

$$D_T - \frac{1}{2}\sigma^2(K, T)K^2D_{KK} = 0. \quad (10)$$

As we will now show, this equation does not hold for the general cases of either arbitrary combinations of  $r$  and  $\nu$ , or state-dependent local volatility functions. It is only true when  $r = \nu$  or when local volatilities are functions of time only. Thus, the transition from spot prices to forward prices does not automatically remove all terms depending on interest rates and dividend yields.

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<sup>1</sup>One could obtain the backward pde in the form given by DFW starting from the dynamics of the forward price under the risk-neutral measure,  $dF_t = \sigma(F, t)F_t d\widehat{W}_t$ . Then, however, the local volatility function of the stochastic process for the stock price,  $S = e^{-(r-\nu)(T-t)}F_t$ , would also be given by  $\sigma(Se^{(r-\nu)(T-t)}, t)$ . The key point thus is that the two local volatility functions for the stock and the forward price have to be identical.

We begin our derivation with the standard forward pde (6) for a plain vanilla call. With (8) and (9) we get

$$\begin{aligned} C_K &= e^{-r(T-t)} D_K \\ C_{KK} &= e^{-r(T-t)} D_{KK} \\ C_T &= \frac{\partial}{\partial T} \left[ D(S e^{(r-\nu)(T-t)}, t; K, T) e^{-r(T-t)} \right] = e^{-r(T-t)} (D_T + D_F(r - \nu)F - rD) \end{aligned}$$

From this we deduce the pde

$$\frac{1}{2} \sigma^2(K, T) K^2 D_{KK} = (r - \nu) K D_K - (r - \nu) D + D_T + (r - \nu) F D_F \quad (11)$$

The same result can be derived using the transition densities and performing the change of variables there (see Appendix B).

Comparing (11) to (10) shows that the two equations are equivalent if and only if

$$(r - \nu) K D_K - (r - \nu) D + (r - \nu) F D_F = 0 \quad (12)$$

It is easy to see that one special case for which (12) holds is when the interest rate  $r$  is equal to the dividend yield  $\nu$ .

However, there is a not so obvious scenario under which (12) is true. Assume that the local volatility function  $\sigma(u, t)$  depends on  $t$  only, i.e.  $\sigma(u, t) = \sigma(t)$ . It is well known that the price of a European call option in this situation is still given by the Black-Scholes formula, with  $\sigma\sqrt{T-t}$  replaced by  $\sqrt{\int_t^T \sigma^2(s) ds}$ ,

$$C(S, t; K, T) = S e^{-\nu(T-t)} N(d_1) - K e^{-r(T-t)} N(d_2)$$

where

$$\begin{aligned} d_1 &= \frac{\ln(S/K) + (r - \nu)(T - t) + 0.5 \int_t^T \sigma^2(s) ds}{\sqrt{\int_t^T \sigma^2(s) ds}} \\ d_2 &= d_1 - \sqrt{\int_t^T \sigma^2(s) ds}. \end{aligned}$$

In terms of partial derivatives the Black-Scholes formula can be written as

$$C(S, t; K, T) = S C_S + K C_K$$

since

$$\begin{aligned} C_S &= e^{-\nu(T-t)} N(d_1) \\ C_K &= -e^{-r(T-t)} N(d_2). \end{aligned}$$

With

$$C_S = e^{-\nu(T-t)} D_F$$

and

$$C_K = e^{-r(T-t)} D_K$$

the Black-Scholes formula becomes the following forward-based expression known as the Black formula

$$\begin{aligned} D(F, t; K, T) &= FN(d_1) - KN(d_2) \\ &= FD_F + KD_K \end{aligned}$$

Multiplying both sides of this equation by  $(r - \nu)$  yields (12), which means that (10) holds.

## 4 Summary

This paper deals with pdes in the theory of contingent claim pricing. Starting from standard forward and backward equations we derive forward and backward pdes for the forward prices of claims written on the forward price of the underlying asset.

In the course of our analysis we find that there are two errors in the pdes proposed by DFW [4]. First, in their backward equation the first argument of the local volatility function is the forward price, whereas it should be the discounted forward price. Furthermore, their forward equation does not hold in the general case of arbitrary interest rates and dividend yields or when the local volatility function is explicitly state-dependent.

## References

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# Appendix

## A Calls on Forwards

**Proposition 1 (Calls on Forwards, Black (1976) [2])** *The price of a call with strike  $K$  and maturity  $T_1$  on a forward contract with maturity  $T_2 > T_1$  is given by*

$$\bar{C}(F, t, T_2; K, T_1) = Fe^{-r(T_1-t)} \tilde{P}_t(F(T_1, T_2) > K) - Ke^{r(T_1-t)} \hat{P}_t(F(T_1, T_2) > K)$$

where  $\tilde{P}$  and  $\hat{P}$  denote the martingale measures obtained by using the modified stock price  $Se^{-\nu(T-t)}$  and the money market account  $B$  as numeraires, respectively.

Note that for  $T_1 = T_2 = T$  this formula is nothing but the well-known Black-Scholes formula for an underlying with continuous dividend yield:

$$\begin{aligned} \bar{C}(F, t; K, T) &= Fe^{-r(T-t)} \tilde{P}(F(T, T) > K) - Ke^{r(T-t)} \hat{P}(F(T, T) > K) \\ &= Se^{-\nu(T-t)} \tilde{P}(S_T > K) - Ke^{r(T-t)} \hat{P}(S_T > K) \\ &= C(S, t; K, T) \end{aligned}$$

### Proof:

Remember that we can value a European call on an underlying  $S$  with continuous dividend yield  $\nu$  by valuing a European call on an underlying with initial stock price  $Se^{-\nu(T-t)}$  at time  $t$  and no dividend payments.

The payoff of the call on the forward at time  $T_1$  is

$$\begin{aligned} \max(F(T_1, T_2) - K; 0) &= F(T_1, T_2)I_{\{F(T_1, T_2) > K\}} - KI_{\{F(T_1, T_2) > K\}} \\ &:= X_{T_1} - Y_{T_1} \end{aligned}$$

where  $I_A$  denotes the indicator function for event  $A$ ,  $X_{T_1} = F(T_1, T_2)I_{\{F(T_1, T_2) > K\}}$ ,  $Y_{T_1} = KI_{\{F(T_1, T_2) > K\}}$ , and  $F(t, T_2)$  is short for  $F(S_t, t, T_2)$ .

We now calculate the price of this option at time  $t$ , using martingale-based valuation techniques. Let  $X_t$  and  $Y_t$  denote the values at time  $t$  of the random cash flows  $X_{T_1}$  and  $Y_{T_1}$ , respectively. For the first term we use the martingale measure  $\tilde{P}$ , obtained by choosing the modified stock  $Se^{-\nu(T-t)}$  as the numeraire, for the second term the martingale measure  $\hat{P}$  with the money market account as the normalizing asset. Since

$$F(T_1, T_2) = S_{T_1}e^{(r-\nu)(T_2-T_1)}$$

we obtain

$$\frac{X_t}{S_t e^{-\nu(T_2-t)}} = \mathbf{E}_t^{\tilde{P}} \left( \frac{X_{T_1}}{S_{T_1} e^{-\nu(T_2-T_1)}} \right)$$

$$\begin{aligned}
&= e^{r(T_2-T_1)} \tilde{P}_t(F(T_1, T_2) > K) \\
\Rightarrow X_t &= F(t, T_2) e^{-r(T_1-t)} \tilde{P}_t(F(T_1, T_2) > K)
\end{aligned} \tag{13}$$

and

$$\begin{aligned}
\frac{Y_t}{e^{rt}} &= \mathbb{E}_t^{\hat{P}} \left( \frac{Y_{T_1}}{e^{rT_1}} \right) \\
&= K e^{-rT_1} \hat{P}_t(F(T_1, T_2) > K) \\
\Rightarrow Y_t &= K e^{-r(T_1-t)} \hat{P}_t(F(T_1, T_2) > K)
\end{aligned} \tag{14}$$

Subtracting (14) from (13) finally yields the proposition.

## B Transformed Forward and Backward Transition Densities

Now we derive the backward and forward pdes for the modified densities. We will see that this leads to the same backward and forward pdes for the forward call, as derived in section 3.

Let  $t, T \in [0, \tau]$ ,  $t \leq T$ . Define

$$\tilde{p}(\tilde{x}, t, y, T) := p(e^{-(r-\nu)(T-t)} \tilde{x}, t, y, T) = p(x, t, y, T)$$

as the modified transition density, where  $x = e^{-(r-\nu)(T-t)} \tilde{x}$ , and  $p(x, t, y, T)$  satisfies both the standard backward pde (4) and the standard forward pde (5). It is obviously true that

$$\int \tilde{p}(\tilde{x}, t, y, T) dy = 1.$$

We want to transform the pdes for  $p(x, t, y, T)$  into analogous pdes for  $\tilde{p}(\tilde{x}, t, y, T)$ . To do so, we need the following partial derivatives:

$$\begin{aligned}
\frac{\partial \tilde{p}(\tilde{x}, t, y, T)}{\partial t} &= \frac{\partial p(x, t, y, T)}{\partial t} + (r - \nu)x \frac{\partial p(x, t, y, T)}{\partial x} \\
\frac{\partial \tilde{p}(\tilde{x}, t, y, T)}{\partial \tilde{x}} &= e^{-(r-\nu)(T-t)} \frac{\partial p(x, t, y, T)}{\partial x} \\
\frac{\partial^2 \tilde{p}(\tilde{x}, t, y, T)}{\partial \tilde{x}^2} &= e^{-2(r-\nu)(T-t)} \frac{\partial^2 p(x, t, y, T)}{\partial x^2}
\end{aligned}$$

From these and (4) we get

$$\begin{aligned}
\frac{\partial \tilde{p}(\tilde{x}, t, y, T)}{\partial t} &= \frac{\partial p(x, t, y, T)}{\partial t} + (r - \nu)x \frac{\partial p(x, t, y, T)}{\partial x} \\
&= -\frac{1}{2} \sigma^2(x, t) x^2 \frac{\partial^2 p(x, t, y, T)}{\partial x^2} \\
&= -\frac{1}{2} \sigma^2(e^{-(r-\nu)(T-t)} \tilde{x}, t) \tilde{x}^2 \frac{\partial^2 \tilde{p}(\tilde{x}, t, y, T)}{\partial \tilde{x}^2}
\end{aligned}$$

Integrating this expression leads to the backward pde (7) for the forward call  $D$ :

$$\frac{\partial D(\tilde{x}, t; K, T)}{\partial t} = -\frac{1}{2} \sigma^2(e^{-(r-\nu)(T-t)} \tilde{x}, t) \tilde{x}^2 \frac{\partial^2 D(\tilde{x}, t; K, T)}{\partial \tilde{x}^2}$$

where

$$D(\tilde{x}, t; K, T) = \hat{\mathbb{E}}_t(\max(F_T - K; 0)) = \int_K^\infty (y - K)\tilde{p}(\tilde{x}, t; y, T)dy$$

For the forward pde we additionally need the following derivatives:

$$\begin{aligned}\frac{\partial \tilde{p}(\tilde{x}, t; y, T)}{\partial T} &= \frac{\partial p(x, t; y, T)}{\partial T} - (r - \nu)x \frac{\partial p(x, t; y, T)}{\partial x} \\ \frac{\partial \tilde{p}(\tilde{x}, t; y, T)}{\partial y} &= \frac{\partial p(x, t; y, T)}{\partial y} \\ \frac{\partial^2 \tilde{p}(\tilde{x}, t; y, T)}{\partial y^2} &= \frac{\partial^2 p(x, t; y, T)}{\partial y^2}\end{aligned}$$

From these and (5) we get

$$\begin{aligned}\frac{\partial \tilde{p}(\tilde{x}, t; y, T)}{\partial T} &= \frac{\partial p(x, t; y, T)}{\partial T} - (r - \nu)x \frac{\partial p(x, t; y, T)}{\partial x} \\ &= \frac{1}{2} \frac{\partial^2}{\partial y^2} [\sigma^2(y, T)y^2 p(x, t; y, T)] - (r - \nu) \frac{\partial}{\partial y} [yp(x, t; y, T)] \\ &\quad - (r - \nu)x \frac{\partial p(x, t; y, T)}{\partial x} \\ &= \frac{1}{2} \frac{\partial^2}{\partial y^2} [\sigma^2(y, T)y^2 \tilde{p}(\tilde{x}, t; y, T)] - (r - \nu) \frac{\partial}{\partial y} [y\tilde{p}(\tilde{x}, t; y, T)] \\ &\quad - (r - \nu)\tilde{x} \frac{\partial \tilde{p}(\tilde{x}, t; y, T)}{\partial \tilde{x}}\end{aligned}\tag{15}$$

To derive the forward pde for a forward call  $D$  we follow the lines of [1].

$$D(\tilde{x}, t; K, T) = \hat{\mathbb{E}}_t(\max(F_T - K; 0)) = \int_K^\infty (y - K)\tilde{p}(\tilde{x}, t; y, T)dy$$

From this it follows that

$$\frac{\partial^2 D(\tilde{x}, t; K, T)}{\partial K^2} = \tilde{p}(\tilde{x}, t; K, T)$$

Plugging this into (15) we get

$$\begin{aligned}\frac{\partial \tilde{p}(\tilde{x}, t; y, T)}{\partial T} &= \frac{\partial^2}{\partial K^2} \frac{\partial D(\tilde{x}, t; K, T)}{\partial T} \\ &= -(r - \nu) \frac{\partial}{\partial K} \left( K \frac{\partial^2 D(\tilde{x}, t; K, T)}{\partial K^2} \right) \\ &\quad + \frac{1}{2} \frac{\partial^2}{\partial K^2} \left( K^2 \sigma^2(K, T) \frac{\partial^2 D(\tilde{x}, t; K, T)}{\partial K^2} \right) \\ &\quad - (r - \nu)\tilde{x} \frac{\partial^2}{\partial K^2} \left( \frac{\partial D(\tilde{x}, t; K, T)}{\partial \tilde{x}} \right)\end{aligned}$$

Integrating this twice with respect to  $K$  leads to

$$\begin{aligned}\frac{\partial D(\tilde{x}, t; K, T)}{\partial T} &= \frac{1}{2} \sigma^2(K, T) K^2 \frac{\partial^2 D(\tilde{x}, t; K, T)}{\partial K^2} - (r - \nu) K \frac{\partial D(\tilde{x}, t; K, T)}{\partial K} \\ &\quad + (r - \nu) D(\tilde{x}, t; K, T) - \frac{\partial D(\tilde{x}, t; K, T)}{\partial \tilde{x}} (r - \nu)\tilde{x}\end{aligned}$$

This is equation (11) with  $\tilde{x} = F$ .